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DISTINCTIVE FEATURES OF THE PROPAGATION OF SOUND WAVES IN A PERFECT GAS AT LOW PRESSURE

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The minimum possible pressure at which a sound wave can propagate in a perfect gas has been found based on the dispersion relation. Using the molecular-kinetic theory it has been shown that this process can occur with such a minimum density when a quarter of a wavelength fits into the distance between molecules. A certain analogy between the propagation of a sound wave under these conditions and the propagation of an electromagnetic-energy quantum has been noted.

The propagation of sound waves at a low pressure is of interest in connection with various acoustic and heat-exchange processes which accompany the motion of aircraft in the upper atmosphere.

When longitudinal mechanical wave packets propagate in gases, dispersion can occur, which leads to the spreading of these packets. The dispersion of a wave is associated with the physical characteristics of a medium and is determined by the parameter

$$\delta = \frac{c_\infty - c_0}{c_0} . \tag{1}$$

If we assume that c_∞ in the gas at a very high frequency is determined by the adiabatic process of compression and rarefaction of the medium in the wave, then to calculate c_∞ we can use the known Laplace formula for the velocity of sound in the gas. As a result of this, we have

$$c_\infty = \sqrt{\gamma \frac{P}{\rho}} . \tag{2}$$

The velocity of sound at the zero frequency must be determined by the isothermal process; therefore, its value can be found from the so-called "erroneous" Newton formula

$$c_0 = \sqrt{\frac{P}{\rho}} . \tag{3}$$

Consequently,

$$c_\infty = c_0 \sqrt{\gamma} . \tag{4}$$

Having substituted formulas (2) and (3) into Eq. (1), we have

$$\delta = \sqrt{\gamma} - 1 . \tag{5}$$

Taking into account the relation between the adiabatic constant and the number of degrees of freedom of a gas molecule $\gamma = 1 + \frac{2}{i}$, we obtain

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$$\delta = \sqrt{1 + \frac{2}{i}} - 1 \approx \frac{1}{i} \,. \tag{6}$$

Thus, the dispersion parameter for the gas is the quantity which is inverse to i. For example, for a perfect monatomic gas at $i = 3$ the dispersion parameter is $\delta = 0.33$. For the diatomic gas at relatively low temperatures ($i = 5$) we have $\delta = 0.2$, while at high temperatures ($i = 7$, the vibrational degrees of freedom are included) we have $\delta = 0.14$. In the case of the polyatomic gas with an asymmetric molecule at relatively low temperatures ($i = 6$), the dispersion parameter is $\delta = 0.17$.

The velocity of sound in the gas during the adiabatic process can be found from the formula

$$c_\infty = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_S} \,, \tag{7}$$

where the subscript S denotes the adiabatic process.

The velocity of sound in the gas during the isothermal process can be found from the formula

$$c_0 = \sqrt{\left(\frac{\partial P}{\partial \rho}\right)_T} \,, \tag{8}$$

where the subscript T denotes the isothermal process.

In the general case [1] we have

$$\frac{\partial P}{\partial \rho} = \frac{c_0^2 - j\omega\tau c_\infty^2}{1 - j\omega\tau} \,. \tag{9}$$

Consequently, the dependence of the velocity of sound on the frequency in the perfect gas has the form

$$c_{\text{r}} = \sqrt{\frac{\partial P}{\partial \rho}} = \sqrt{\frac{c_0^2 - j\omega\tau c_\infty^2}{1 - j\omega\tau}} = \sqrt{c_0^2 \frac{(1 - j\omega\tau\gamma)}{1 - j\omega\tau}} = \sqrt{\frac{P}{\rho} \frac{(1 - j\omega\tau\gamma)}{1 - j\omega\tau}} \,. \tag{10}$$

The real value of the velocity of sound is the real part of Eq. (10), which, with the use of Eq. (4), is represented as

$$c_{\text{r}} = \sqrt{c_0^2 \frac{1 + \omega^2\tau^2\gamma}{1 + \omega^2\tau^2}} \,. \tag{11}$$

For a moving compressible gas the energy dissipation is determined by two parameters: the internal friction in the gas and its volumetric expansion or compression. Therefore, apart from the usual viscosity μ characterizing the internal friction, we introduce the notion of the second viscosity ζ_0, which characterizes the dissipation of energy in compression or expansion of the moving gas.

In accordance with [1], the second viscosity depends on the sound frequency. Therefore, the second viscosity during the isothermal process will be considered to be the characteristic of the gas; the modulus of the second viscosity is equal to

$$\zeta_0 = \rho\tau \,(c_\infty^2 - c_0^2) = \rho\tau c_0^2 \,(\gamma - 1) = \frac{2\tau P}{i} \,, \tag{12}$$

while in sign it is negative.

In accordance with the Stokes hypothesis [2], during the isothermal process we have

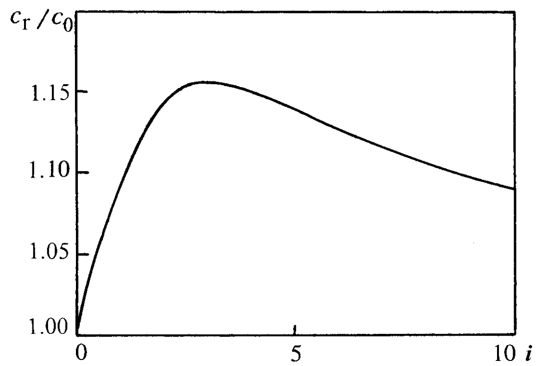

Fig. 1. Dependence of the relative velocity of sound in a rarefied gas on the number of degrees of freedom of the gas molecules.

$$\zeta_0 = -\frac{2}{3}\mu . \tag{13}$$

Only with condition (13) are the normal stresses in the gas equivalent to the pressure in the gas with the opposite sign.

Determining the relaxation time τ from Eq. (12) and taking into account Eq. (13), we have

$$\tau = \frac{\mu i}{3P} . \tag{14}$$

Substitution of Eq. (14) into Eq. (11) gives

$$\frac{c_r}{c_0} = \sqrt{\left(1 + \left(\frac{\mu\omega}{3P}\right)^2 i\,(i+2)\right) \Big/ \left(1 + \left(\frac{\mu\omega}{3P}\right)^2 i^2\right)} . \tag{15}$$

The graph of the function $c_r(i)/c_0$ plotted for $P/(\mu\omega) = 1$ is shown in Fig. 1, from which it follows that at a certain number of degrees of freedom there is a maximum in the wave velocity. Determining the first derivative of Eq. (15) with respect to i and equating it to zero, we obtain that the maximum of the velocity is observed for

$$i_m = \frac{3P}{\mu\omega} . \tag{16}$$

For the perfect monoatomic gas we have $i = 3$; therefore, $P/(\mu\omega) = 1$. The graph is plotted precisely for this condition.

Since the gas molecules cannot have degrees of freedom less than three, the gas pressure entering into formula (16) is the minimum possible at which the sound wave propagates. Thus, in Fig. 1 only the descending portion of the curve has a physical meaning.

For the perfect monatomic gas the relation between the minimum pressure and the sound frequency has the form

$$P = \mu\omega . \tag{17}$$

Analysis of Eq. (17), performed on the basis of the molecular-kinetic theory, has confirmed the fact that the pressure in the formula is the minimum possible. Taking into account the relation between the pressure and the root-mean-square velocity of molecular motion and also between the viscosity and the microparameters of the substance, we find

$$\frac{1}{3}\rho V_{r\text{-}m\text{-}s}^2 = \frac{1}{3}\rho V_{m.v}\bar{\lambda}\omega , \tag{18}$$

where $V_{r-m-s} = \sqrt{3kT/m}$ and $V_{m.v} = \sqrt{8kT/(\pi m)}$.

The rearrangements carried out give

$$\frac{3}{16} V_{m.v} = \bar{\lambda}\nu . \tag{19}$$

Taking in accordance with [3] that $V_{m.v} = \sqrt{\frac{3}{\gamma}} c_\infty = \frac{3}{\sqrt{5}} c_\infty = \frac{3}{\sqrt{5}} \lambda\nu$, we determine that

$$\lambda = \frac{16\sqrt{5}}{9} \bar{\lambda} \approx 4\bar{\lambda} . \tag{20}$$

It is evident from the relation (20) obtained that the wave in the gas can propagate with such a minimum density when a quarter of the wavelength fits into the distance between the molecules. It is obvious that at a lower density the wave cannot appear. Therefore, Eq. (17) makes it possible to calculate the minimum pressure in the perfect monatomic gas at which the sound wave can appear. A small deviation in Eq. (20) from the exact value (equal to four) is apparently associated with the fact that formula (18) must involve the mean product $\overline{\lambda V}$ and not the product of the mean quantities $V_{m.v}\bar{\lambda}$ [4].

Expression (17) can be considered as the equation for the minimum possible energy of thermal motion of molecules $\bar{E}$ on the front of the sound wave occurring due to the compression of the medium in the wave and transferred by this wave. In this case, the above equation becomes fully identical, in its meaning, with the equation for the quantum energy in quantum mechanics $E = h\nu$.

Indeed, from Eq. (17) it follows that $P = 2n\bar{E}/3 = \mu\omega$. Thus, for the perfect gas we have

$$\bar{E} = \frac{3\pi\mu}{n} \nu = h_{sn}\nu , \tag{21}$$

where $h_{sn} = 3\pi\mu/n$. Formula (21) determines the energy of the so-called "acoustic quantum." This quantum cannot possess a lower energy, since the number concentration of molecules is insufficient for organizing the wave process.

The analogy with an electromagnetic quantum becomes even more transparent if we note that

$$h_{sn} = \frac{3\pi\mu}{n} = \frac{\pi\rho\bar{\lambda}V_{m.v}}{n} = \pi m\bar{\lambda}V_{m.v} = \frac{3\pi m\bar{\lambda}}{\sqrt{5}} c_\infty . \tag{22}$$

For the Planck constant we can obtain the formula

$$h = 2\pi m_{el} l_C c , \tag{23}$$

where c is the velocity of light in vacuum.

However, this analogy, just as any other, has its own limitations. First, in contrast to the Planck constant, its sound analog is not constant and depends on the pressure, temperature, and number of the degrees of freedom of the gas molecule. Second, the existence of the minimum possible energy of the electromagnetic quantum is based on quite different processes associated with the properties of the vacuum rather than on the transfer of the minimum energy of thermal motion by the wave, even though a certain external generality of the basic principles can be observed.

In closing, we evaluate the numerical value of h_{sn} for air with $i = 5$. The minimum air pressure at which sound propagates can be calculated from the formula $P = 5\mu\omega/3$. The dynamic air viscosity μ at the given temperature depends only slightly on the pressure. This is due to the constancy of the product of the density of the gas and the mean free path of the molecule which enters into the expression for the gas viscosity. Actually, the formula $P = 5\mu\omega/3$ determines the largest height above the earth's surface where the sound wave of the given frequency can propagate.

In accordance with [5], the air pressure becomes lower than that calculated from the formula $P = 5\mu\omega/3$ at a height of about 95 km above the earth's surface. As an example, the frequency of the sound wave is here taken to be 1000 Hz. The air viscosity at this height is $\mu = 13.2\cdot10^{-6}$ Pa·sec [5]. Sound waves with frequencies higher than 1000 Hz will not propagate at the indicated height since this requires a higher air pressure. Taking the frequency of the sound wave to be 1000 Hz, we find the air pressure at this height $P = 0.14$ Pa, which corresponds to experimental data [5]. At a height of 95 km, the air temperature is $T = 200$ K [5]. The number of molecules per unit volume of the gas at this pressure and temperature is $n = P/(kT) = 5.07\cdot10^{19}$ m^{-3}. Consequently, for air as for the diatomic gas we have $h_{sn} = i^2\pi\mu/(3n) = 25\pi\mu/(3n) = 6.81\cdot10^{-24}$ J·sec, while the Planck constant is ten orders of magnitude smaller than its acoustic analog.

Thus, as an aircraft approaches the upper atmosphere, the process of transfer of acoustic signals is impaired, since it occurs due to the "acoustic quanta" of a lower frequency, which is explained by a decrease in the upper frequency limit of longitudinal mechanical waves appearing in the atmosphere.

NOTATION

c_∞, c_0, and c_r, velocities of sound at very high (infinite), very low (zero), and arbitrary frequencies; δ, dispersion parameter; γ, adiabatic exponent; P, mean pressure of the gas; ρ, gas density; i, number of degrees of freedom of a gas molecule; T, absolute temperature of the gas; j, imaginary unit; ω, cyclic frequency of the sound wave; τ, relaxation time of the process; μ, dynamic molecular viscosity of the gas; ζ_0, second viscosity during the isothermal process; $\bar{\lambda}$, mean free path of the gas molecules; V_{r-m-s}, $V_{m.v}$, and V, root-mean-square, arithmetic mean, and arbitrary velocities of the gas molecules; k, Boltzmann constant; m, mass of a single gas molecule; ν, wave frequency; λ, wavelength in the gas; $\bar{E}$, mean energy of thermal motion of the gas molecules on the sound-wave front; h and h_{sn}, Planck constant and its sound analog; n, number of molecules per unit volume of the gas; m_{el}, electron mass; l_C, Compton wavelength of the electron. Subscripts: r, real; m, maximum; sn, sound; C, Compton; r-m-s and m.v, root-mean-square and mean values; el, electron.

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